

Noncommutativity in the early Universe

G. Oliveira-Neto, M. Silva de Oliveira*, G. A. Monerat
and E. V. Corrêa Silva†

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Abstract

In the present work, we study the noncommutative version of a quantum cosmology model. The model has a Friedmann-Robertson-Walker geometry, the matter content is a radiative perfect fluid and the spatial sections have zero constant curvature. In this model the scale factor takes values in a bounded domain. Therefore, its quantum mechanical version has a discrete energy spectrum. We compute the discrete energy spectrum and the corresponding eigenfunctions. The energies depend on a noncommutative parameter β . We compute the scale factor expected value ($\langle a \rangle$) for several values of β . For all of them, $\langle a \rangle$ oscillates between maxima and minima values and never vanishes. It gives an initial indication that those models are free from singularities, at the quantum level. The $\langle a \rangle$ behavior, for the present model, is a drastic modification of the $\langle a \rangle$ behavior in the corresponding commutative version of the present model. There, $\langle a \rangle$ grows without limits with the time variable. Therefore, if the present model may represent the early stages of the Universe, the results of the present paper give an indication that $\langle a \rangle$ may have been, initially, bounded due to noncommutativity.

*The first two authors are from: Departamento de Física, Instituto de Ciências Exatas, Universidade Federal de Juiz de Fora, CEP 36036-330 - Juiz de Fora, MG, Brazil. gilneto@fisica.ufjf.br

†The last two authors are from: Faculdade de Tecnologia, Universidade do Estado do Rio de Janeiro, Rodovia Presidente Dutra, Km 298, Pólo Industrial, CEP 27537-000, Resende-RJ, Brazil. monerat@uerj.br and evasquez@uerj.br

Noncommutative ideas were first introduced, a long time ago, by Snyder [1, 2]. There, the noncommutativity was imposed between the spacetime coordinates and his main motivation was to eliminate the divergences in quantum field theory. Recently, the interest in those ideas of noncommutativity between spacetime coordinates were renewed due to some important results obtained in superstring, membrane and M -theories [3, 4, 5, 6, 7]. For more information on those important results we refer to the reviews [8, 9]. Since then, noncommutativity has been applied to many other physical systems, such as: quantum harmonic oscillator [10, 11, 12], hydrogen atom [13], quantum Hall effect [14, 15, 16], Einstein's gravity theory [17, 18, 19], cosmology [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31], black hole physics [32, 33, 35, 34, 36, 37, 38, 39, 40, 41], quantum cosmology [42, 43, 44, 45], to name only but a few. For a more complete list of references see [46].

One important arena where noncommutative (NC) ideas may play an important role is cosmology. In the early stages of its evolution, the Universe may have had very different properties than the ones it has today. Among those properties some physicists believe that the spacetime coordinates were subjected to a noncommutative algebra. Inspired by these ideas some researchers have considered such NC models in quantum cosmology [42, 43, 44, 45]. It is also possible that some residual NC contribution may have survived in later stages of our Universe. Based on these ideas some researchers have proposed some NC models in classical cosmology in order to explain some intriguing results observed by WMAP. Such as a running spectral index of the scalar fluctuations and an anomalously low quadrupole of CMB angular power spectrum [21, 22, 23, 24, 25]. Another relevant application of the NC ideas in semi-classical and classical cosmology is the attempt to explain the present accelerated expansion of our Universe [28, 29, 30, 31].

In Ref. [31], several different noncommutative classical Friedmann Robertson Walker (FRW) cosmological models were studied. There, they work in the Schutz's variational formalism [47, 48] and use the Hamiltonian formalism. Therefore, the phase space of those models is given by the following canonical variables and conjugated momenta: $\{a, p_a, T, p_T\}$, where a is the scale factor, T is a time variable associated to the fluid and p_a and p_T are, respectively, their conjugated momenta. They consider a noncommutativity between the two momenta p_a and p_T . In subsection 4.3, page 15 of Ref. [31], they consider a model, that may represent the early stages of the Universe, with flat spatial sections and a radiative perfect fluid. For a positive noncommutative parameter, they show that the scale factor behavior is dras-

tically modified with respect to the corresponding commutative version of the model. For the commutative version, the scale factor grows and eventually goes to infinity when the time goes to infinity, following the rule (in the gauge $N = 1$) [49],

$$a(t) = \sqrt{\sqrt{4E/3}t + a_0^2}, \quad (1)$$

where t is the coordinate time, E is the radiation energy and a_0 is the scale factor value for $t = 0$. On the other hand, in the noncommutative version the scale factor remains bounded. If the Universe starts expanding from a small scale factor value, after a finite time it reaches a maximum value and then contracts to the singularity. In order to see that behavior, consider equations (4.2), (4.3) and (4.11) of Ref. [31]. From them, we obtain the following equations describing the scale factor dynamics (in the gauge $N = 1$),

$$\dot{a}^2 + \frac{\beta}{3a} - \frac{E}{3a^2} = 0, \quad (2)$$

$$2\ddot{a}a + \dot{a}^2 + \frac{E}{3a^2} = 0, \quad (3)$$

where the dot means derivative with respect to the coordinate time t , β is the noncommutative parameter and we have used the notation of the present work to name the fluid energy (E). If one chooses $\beta = 0.1$, $E = 1.336713605$, $a(t = 0) = 0.1$ and $\dot{a}(t = 0) = 6.650096751$ and solve Eqs. (2-3), one obtains the result shown in Figure 1. In that figure, the scale factor stops before reaching the singularity due to numerical limitations. It means that, if this model may represent the early stages of the Universe, it gives an indication that the scale factor may have been, initially, bounded due to noncommutativity. Since, quantum cosmology is more appropriate to explain the initial stages of the Universe, than classical cosmology, we have decided investigating if that important indication is still true, at the quantum level.

In the present work, we study the quantum cosmology version of the noncommutative model described above. The noncommutativity, at the quantum level, we are about to propose will be between the canonically conjugated momenta to the scale factor and the radiative perfect fluid, following the choice made, at the classical level, by the authors of Ref. [31]. Since these variables are functions of the time coordinate t , this procedure is a generalization of the typical noncommutativity between usual spatial coordinates. The noncommutativity between those types of phase space variables have already

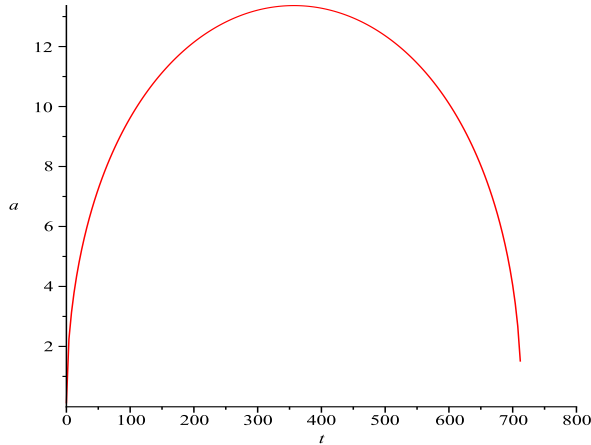


Figure 1: $a(t)$ as a function of t , for $\beta = 0.1$, $E = 1.336713605$, $a(t = 0) = 0.1$ and $\dot{a}(t = 0) = 6.650096751$.

been proposed in the literature. At the quantum level in Refs. [42, 43, 44, 45] and at the semi-classical and classical levels in Refs. [28, 29, 30, 31]. We quantize the model and obtain the appropriate Wheeler-DeWitt equation. In this model the scale factor takes values in a bounded domain. Therefore, its quantum mechanical version has a discrete energy spectrum. We compute the discrete energy spectrum and the corresponding eigenfunctions. The energies grow with a noncommutative parameter β . We compute the scale factor expected value ($\langle a \rangle$) for several values of β . For all of them, $\langle a \rangle$ oscillates between maxima and minima values and never vanishes. It gives an initial indication that those models are free from singularities, at the quantum level. We observe that, $\langle a \rangle$ grows with the decrease of β . We also observe that, the smaller the value of β , the greater is the interval where $\langle a \rangle$ takes values. All these results confirm, at the quantum level, the results obtained in Ref. [31], for the scale factor, at the classical level.

The FRW cosmological models are characterized by the scale factor $a(t)$ and have the following line element,

$$ds^2 = -N^2(t)dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \quad (4)$$

where $d\Omega^2$ is the line element of the two-dimensional sphere with unitary radius, $N(t)$ is the lapse function and k gives the type of constant curvature

of the spatial sections. Here, we are considering the case with zero curvature $k = 0$ and we are using the natural unit system, where $\hbar = c = G = 1$. The matter content of the model is represented by a perfect fluid with four-velocity $U^\mu = \delta_0^\mu$ in the comoving coordinate system used. The total energy-momentum tensor is given by,

$$T_{\mu,\nu} = (\rho + p)U_\mu U_\nu - pg_{\mu,\nu}, \quad (5)$$

where ρ and p are the energy density and pressure of the fluid, respectively. Here, we assume that $p = \rho/3$, which is the equation of state for radiation. This choice may be considered as a first approximation to treat the matter content of the early Universe and it was made as a matter of simplicity. It is clear that a more complete treatment should describe the radiation, present in the primordial Universe, in terms of the electromagnetic field.

From the metric (4) and the energy momentum tensor (5), one may write the total Hamiltonian of the present model ($N\mathcal{H}$), where N is the lapse function and \mathcal{H} is the superhamiltonian constraint. It is given by [48],

$$N\mathcal{H} = -\frac{p_a^2}{12} + p_T, \quad (6)$$

where p_a and p_T are the momenta canonically conjugated to a and T , the latter being the canonical variable associated to the fluid [47, 48]. Here, we are working in the conformal gauge, where $N = a$. The commutative version of the present model was first treated in Ref. [50].

We wish to quantize the model following the Dirac formalism for quantizing constrained systems [51]. First we introduce a wave-function which is a function of the canonical variables a and T ,

$$\Psi = \Psi(a, T). \quad (7)$$

Then, we impose the appropriate commutators between the operators a and T and their conjugate momenta p_a and p_T . Working in the Schrödinger picture, the operators a and T are simply multiplication operators, while their conjugate momenta are represented by the differential operators,

$$p_a \rightarrow -i\frac{\partial}{\partial a}, \quad p_T \rightarrow -i\frac{\partial}{\partial T}. \quad (8)$$

Finally, we demand that the operator corresponding to $N\mathcal{H}$ annihilate the wave-function Ψ , which leads to the Wheeler-DeWitt equation,

$$\frac{1}{12}\frac{\partial^2}{\partial a^2}\Psi(a, \tau) = -i\frac{\partial}{\partial \tau}\Psi(a, \tau), \quad (9)$$

where the new variable $\tau = -T$ has been introduced. This is the Schrödinger equation of an one dimensional free particle restricted to the positive domain of the variable.

The operator $N\hat{\mathcal{H}}$ is self-adjoint [50] with respect to the internal product,

$$(\Psi, \Phi) = \int_0^\infty da \, \Psi(a, \tau)^* \Phi(a, \tau), \quad (10)$$

if the wave functions are restricted to the set of those satisfying either $\Psi(0, \tau) = 0$ or $\Psi'(0, \tau) = 0$, where the prime \prime means the partial derivative with respect to a . Here, we consider wave functions satisfying the former type of boundary condition and we also demand that they vanish when a goes to ∞ . For the boundary conditions mentioned above, the author of Ref. [50] solved Eq. (9) and used that solution to compute $\langle a \rangle$ for that model. He obtained, for the boundary condition $\Psi(0, \tau) = 0$ (in the gauge $N = a$),

$$\langle a \rangle = \frac{1}{6} \sqrt{\frac{2}{\pi\sigma}} \sqrt{\sigma^2 \tau^2 + (6 - p\tau)^2}, \quad (11)$$

where σ is a positive number, p is a real number and τ is the time variable. Therefore, $\langle a \rangle$ starts from a nonzero value and when τ grows it also grows. Eventually, when $\tau \rightarrow \infty$ also $\langle a \rangle \rightarrow \infty$. From Eq. (11), in that limit, $\langle a \rangle \propto \tau$.

In order to introduce the noncommutativity in the present model, we shall follow the prescription used in Refs. [42, 43, 44, 45]. In those models the noncommutativity was between the operators associated to the canonical variables. Here, it will be between the two operators p_a and p_T , associated to the canonical momenta,

$$[\check{p}_a, \check{p}_T] = i\beta, \quad (12)$$

where \check{p}_a and \check{p}_T are the noncommutative version of the operators. We follow, here, the choice compatible, at the quantum level, with the one made by the authors of Ref. [31], at the classical level. This noncommutativity between those operators can be taken to functions that depend on the noncommutative version of those operators with the aid of the Moyal product [52, 53, 7, 8]. Consider two functions of \check{a} and \check{T} , let's say, f and g . Then, the Moyal product between those two function is given by: $f(\check{a}, \check{T}) \star g(\check{a}, \check{T}) = f(\check{a}, \check{T}) \exp \left[(i\theta/2) (\overrightarrow{\partial_{\check{a}}} \overrightarrow{\partial_{\check{T}}} - \overleftarrow{\partial_{\check{T}}} \overleftarrow{\partial_{\check{a}}}) \right] g(\check{a}, \check{T})$.

Using the Moyal product, we may adopt the following Wheeler-DeWitt equation for the noncommutative version of the present model,

$$\left[\frac{1}{12} \check{p}_a \star \check{p}_a - \check{p}_T \right] \star \Psi(\check{a}, \check{T}) = 0. \quad (13)$$

It is possible to rewrite the Wheeler-DeWitt equation (13) in terms of the commutative version of the operators \check{p}_a and \check{p}_T and the ordinary product of functions. In order to do that, we must initially introduce the following transformation between the noncommutative and the commutative operators,

$$\begin{aligned} \check{p}_a &= p_a + \beta T, \\ \check{p}_T &= p_T, \end{aligned} \quad (14)$$

and the transformations of the other noncommutative variables are trivial: $\check{a} = a$ and $\check{T} = T$. We follow, here, the choice compatible, at the quantum level, with the one made by the authors of Ref. [31], at the classical level. Then, we may write the commutative version of the Wheeler-DeWitt equation (13), to first order in the commutative parameter β , in the Schrödinger picture as,

$$\frac{1}{12} \frac{\partial^2 \Psi(a, \tau)}{\partial a^2} - \frac{i}{6} \beta \tau \frac{\partial \Psi(a, \tau)}{\partial a} = -i \frac{\partial \Psi(a, \tau)}{\partial \tau}, \quad (15)$$

where we have made the following transformation $\tau \rightarrow -T$, in the same way the author of Ref. [50], so that, we may compare the $\langle a \rangle$ computed using our solution with Eq. (11). For a vanishing β this equation reduces to the commutative Schrödinger equation (9), described above.

In order to solve this equation, satisfying the boundary conditions: $\Psi(0, \tau) = 0$ and $\lim_{a \rightarrow \infty} \Psi(a, \tau) \rightarrow 0$, we start imposing that the wave function $\Psi(a, \tau)$ has the following form,

$$\Psi(a, \tau) = e^{i\beta a \tau} e^{-iE\tau} A(a). \quad (16)$$

Introducing this ansatz in Eq. (15), we obtain, to first order in β , the eigenvalue equation,

$$\frac{d^2 A(a)}{da^2} - (12\beta a - 12E)A(a) = 0, \quad (17)$$

where E is the eigenvalue and it is associated with the fluid energy.

The solutions to this equation are the Airy functions,

$$A(a) = c_1 Ai\left(\frac{12\theta a - 12E}{(12\beta)^{2/3}}\right) + c_2 Bi\left(\frac{12\theta a - 12E}{(12\beta)^{2/3}}\right).$$

The Airy functions Bi grow up exponentially when $a \rightarrow \infty$. In order to eliminate this undesirable behavior, we put $c_2 = 0$. Then, the energy eigenfunctions for our model are,

$$A(a) = c_1 Ai\left(\frac{12\beta a - 12E}{(12\beta)^{2/3}}\right). \quad (18)$$

If we introduce the boundary condition that $A(a=0) = 0$, we find from Eq. (18) the energy eigenvalues with the following expression,

$$E_n = \frac{1}{12}(12\beta)^{2/3}\alpha_n \quad (19)$$

where α_n is positive and is the zero of order n of the Airy function Ai . It is clear from this equation that the energy eigenvalues grow with β .

The most general expression of $\Psi(a, \tau)$ Eq. (16), which is a solution to Eq. (15), is a linear combination of the eigenfunctions $A_n(a)$, Eq. (18), taking in account the energy eigenvalues Eq. (19), combined with the exponential factor present in Eq. (16), for a given β value.

$$\Psi(a, \tau) = e^{i\beta a \tau/2} \sum_{n=0}^N C_n Ai\left(\frac{12\beta a - 12E_n}{(12\beta)^{2/3}}\right) \exp(-iE_n \tau). \quad (20)$$

In order to build a wave packet from Eq. (20) one has, initially, to fix the values of β and the number N of energy eigenfunctions contributing to the sum. After that, one has to compute the N energy eigenvalues E_n Eq. (19), with the aid of the first N zeros (α_n) of the Airy function Ai . Also, one has to fix the values of the N coefficients C_n . Finally, one has to introduce the explicit values of all those quantities in Eq. (20) and perform the indicated sum. The time evolution of the wave packets built from Eq. (20) shows that they are null not only at the origin but they are asymptotically null at infinity as well. In the region near $a = 0$ these packets present strong oscillations, which decrease as a increases.

Now, we would like to verify that the wavefunction and quantities computed with it are well defined. In order to do that, we shall compute, the

scale factor expected value, $\langle a \rangle$, for different values of β . First of all, let us choose $N = 20$ and $C_n = 1$, for all n , in Eq. (20). Next, we compute the eigenvalues, E_n for those 21 eigenfunctions, with the aid of Eq. (19). In order to do that we must choose the values of β and α_n . In the present situation, we shall choose several different values of β . Finally, we must compute the scale factor expected value, using the following expression,

$$\langle a \rangle (\tau) = \frac{\int_0^\infty a |\Psi(a, \tau)|^2 da}{\int_0^\infty |\Psi(a, \tau)|^2 da}. \quad (21)$$

After computing $\langle a \rangle$ for several different values of β and various τ intervals, we noticed that this quantity oscillates between maxima and minima values and never vanishes. It gives an initial indication that those models are free from singularities, at the quantum level. We also noticed that, $\langle a \rangle$ grows with the decrease of β and the smaller the value of β , the greater is the interval where $\langle a \rangle$ takes values. Therefore, we notice that the introduction of the noncommutativity represented by Eq. (12) modified in an important way the commutative version of the model. In the commutative version of the model the scale factor expected value takes values in an unbounded domain. It expands as the function of τ given by Eq. (11). On the other hand, in the noncommutative version of the model the scale factor mean value takes values in a bounded domain and is periodic in τ . The commutative version of the model may be obtained from the noncommutative one by taking the limit when $\beta \rightarrow 0$. The above results show clearly that limit from one version to the other. If we start decreasing the value of β the scale factor expected value will oscillate in an ever increasing domain until we set $\beta \rightarrow 0$. At that limit Eq. (15) reduces to Eq. (9) and the scale factor expected value will grow without limits, as the function of τ given by Eq. (11).

Figure 2 shows $\langle a \rangle$ for $\beta = 0.1$ and the τ interval $0 \leq \tau \leq 10000$. It is clear from that figure that $\langle a \rangle$ oscillates between maxima and minima values and never vanishes. Figure 3, shows $\langle a \rangle$ computed under the same conditions used to compute this quantity in Figure 2, except that in this new figure we considered $\beta = 0.0000001$. From Figure 3, it is clear that $\langle a \rangle$ oscillates over a much higher amplitude than in the previous figure and also oscillates during a wider τ interval, in order to have a similar shape to $\langle a \rangle$ in Figure 2. Those behaviors may be understood by the fact that the potential barrier, that confines the scale factor, grows with β . Therefore, as β increases the scale factor is forced to oscillate in an ever decreasing region. Figure 4 shows $\langle a \rangle$ as a function of τ , for both commutative and noncommutative models. We fixed

$\sigma = 0.0008907244952$ ($p = 0.001$) in Eq. (11), such that, both commutative and noncommutative $\langle a \rangle$ have the same initial value at $\tau = 0$ (13.36713605) and for the noncommutative model we chose $\beta = 0.1$.

The above results, which are exemplified by Figures 2 and 3, indicate that, also at the quantum level, the scale factor may have been, initially, bounded, due to the presence of noncommutativity. The difference between the scale factor behavior at the classical and the quantum levels is the fact that, in the quantum noncommutative version of the model the scale factor expected value never goes to zero (the singularity). It oscillates between maxima and minima values, as shown in Figures 2 and 3. On the other hand, in the classical noncommutative version the scale factor oscillates between a minimum and a maximum value and then goes to zero, as shown in Figure 1. It is important to investigate if those behaviors generalize for different types of noncommutativity and other models, that may represent the early stages of the Universe. In this sense, we may mention that in a previous work [45] the authors quantized a noncommutative FRW model with $k = 1$ and radiation. For a noncommutativity between the scale factor (a) and the variable associated to the radiative fluid (T), they showed that it is not possible to solve the appropriate Wheeler-DeWitt equation and find a wavefunction, with the boundary condition $\Psi(0, T) = 0$.

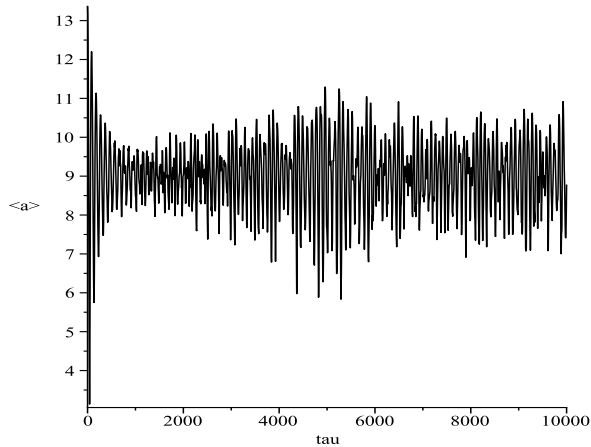


Figure 2: $\langle a \rangle$ for $\beta = 0.1$ and the time interval $0 \leq \tau \leq 10000$.

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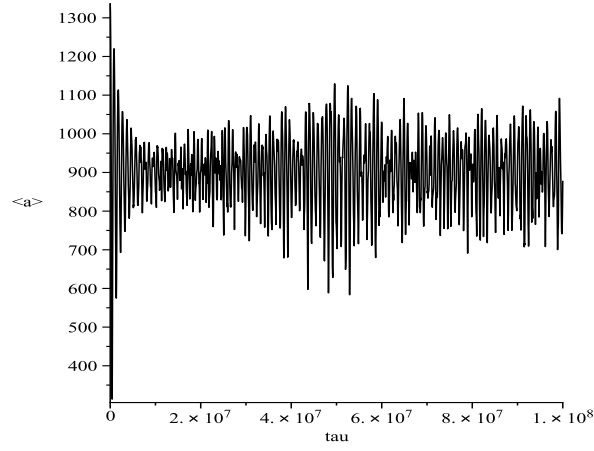


Figure 3: $\langle a \rangle$ for $\beta = 0.0000001$ and the time interval $0 \leq \tau \leq 1 \times 10^8$.

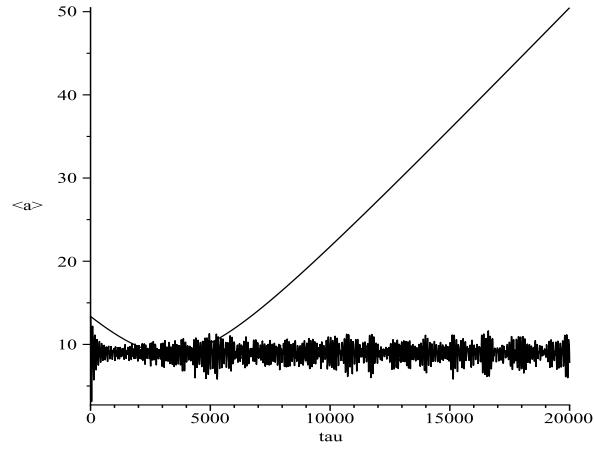


Figure 4: $\langle a \rangle$ as a function of τ for both commutative ($\beta = 0$) and non-commutative ($\beta = 0.1$) cases, in the time interval $0 \leq \tau \leq 20000$. They both start at the same value 13.36713605. $\langle a \rangle$ for the noncommutative case oscillates between maxima and minima values and never goes to zero.

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